Overview of the conventional Gross-Neveu phase diagram and recent noteworthy reports (Review)

Sakura Itatani, M1 from SOKENDAI

Sep. 20th, SSI2023 @Hiroshima University

Self-introduction and today's talk

Name: Sakura Itatani (板谷さくら)

Mar., 2023 : bachelor's degree (Hiroshima U.)

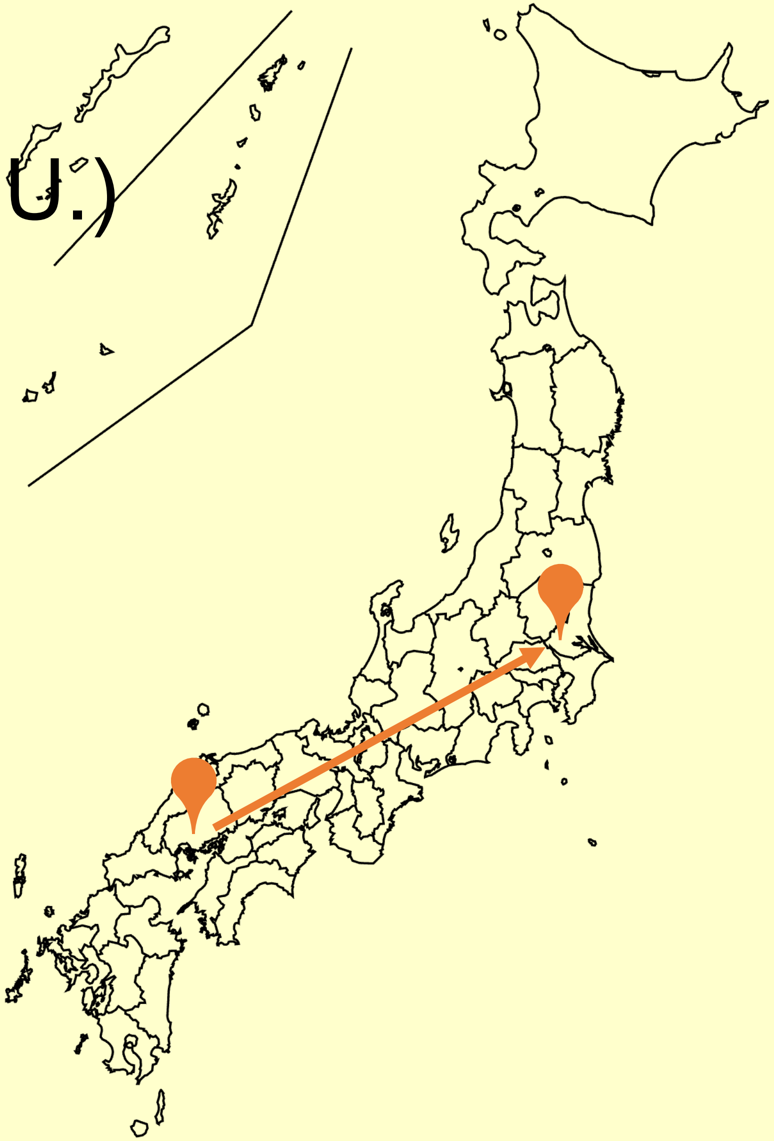
- supervisor

  Ken-Ichi Ishikawa san

- graduate thesis ⊂ today's talk

  "A review of chiral phase transition in the 1+1 dimensional Gross-Neveu model"

Apr., 2023~ : M1 in SOKENDAI (KEK)



Contents

I. The 1+1 dimensional Gross-Neveu model (~ 10 min.)

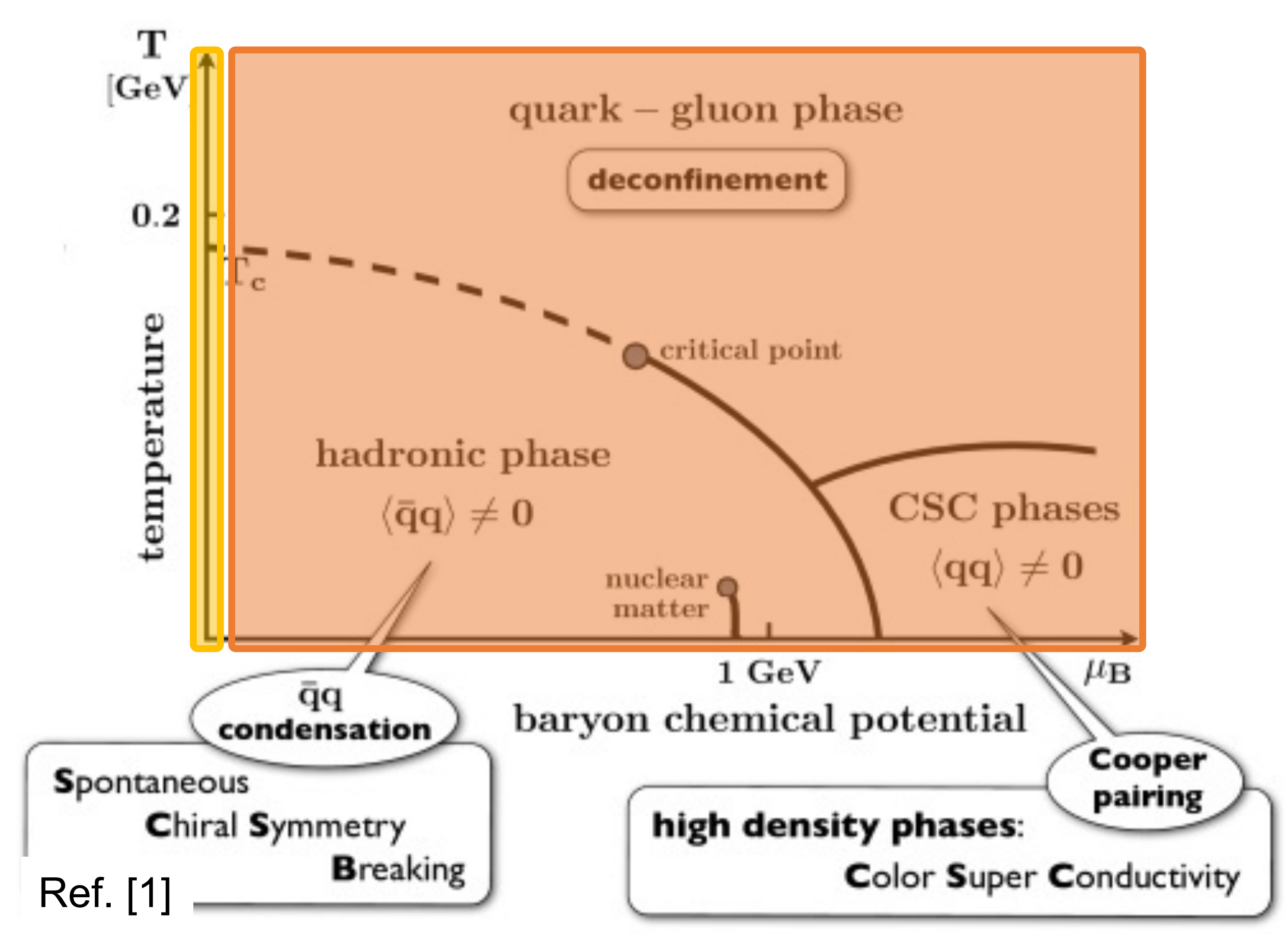
- Introduction
- Scheme for the GN phase diagram
- Calculation results of the phase diagram

II. Phase structures in higher dimensional theories (~ 5 min.)

- Probes for the phase structures

The theory of the strong interaction

Quantum Chromodynamics (QCD)



- The early universe
- Hadronic physics
- Neutron strars etc.

low density

- Nonperturbative
- Lattice QCD

finite density

✗ Lattice QCD (sign problem)

→ toy models

- Nambu-Jona-Lasinio (1961)
- **Gross-Neveu (1974)**

QCD v.s. the Gross-Neveu model

- QCD Lagrangian

(Minkowski spacetime)

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma^{\mu}D_{\mu} - M)\psi - \frac{1}{4}G^{a}_{\mu\nu}G^{\mu\nu a}$$

$$D_{\mu} = \partial_{\mu} - igA^{a}_{\mu}T^{a}$$

- $SU(3)$ gauge symmetry
- Asymptotic freedom
- continuous chiral symmetry ($M \to 0$)

- 1+1 dim. Gross-Neveu model Ref.[2]

(Euclidean spacetime)

$$\mathcal{L}_{GN} = \bar{\psi}(i\gamma_{\mu}\partial_{\mu})\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

- $SU(3)$ gauge symmetry

✓ Asymptotic freedom

✓ Z_2 symmetry (Z_2: $\psi \to i\gamma^5\psi,\ \bar{\psi} \to i\bar{\psi}\gamma^5$)

We can draw the phase diagram of 1+1 dim. Gross-Neveu model by observing SSB of chiral Z_2 symmetry.

Scheme for the GN phase diagram Ref.[3]

1. order parameter

$$\langle\bar{\psi}\psi(x)\rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{-S_{GN}} \ \bar{\psi}\psi$$

$$S_{GN} = \int \mathrm{d}^2x \, \mathcal{L}_{GN}$$

$\mathcal{L}_{GN}$ includes $(\bar{\psi}\psi)^2$ $\Longrightarrow$ not integrable over $\mathcal{D}\bar{\psi}\mathcal{D}\psi$

2. introduce an auxiliary field σ

$$S_\sigma = \int \mathrm{d}^2x \left(\mathcal{L}_{GN} + \frac{N_f}{2g^2}\sigma^2\right)$$

$$\to \int \mathrm{d}^2x \left(\bar{\psi}i(\gamma_\mu\partial_\mu + \sigma)\psi + \frac{N_f}{2g^2}\sigma^2\right)$$

physics does not change: $(const.)\times Z$

3. Ward-Takahashi identity

$$\langle\bar{\psi}\psi(x)\rangle = i\frac{N_f}{g^2}\langle\sigma(x)\rangle$$

- translational invariance of PI measures

$$\mathcal{D}\sigma = \mathcal{D}(\sigma + a)$$

new order parameter: $\langle\sigma(x)\rangle$

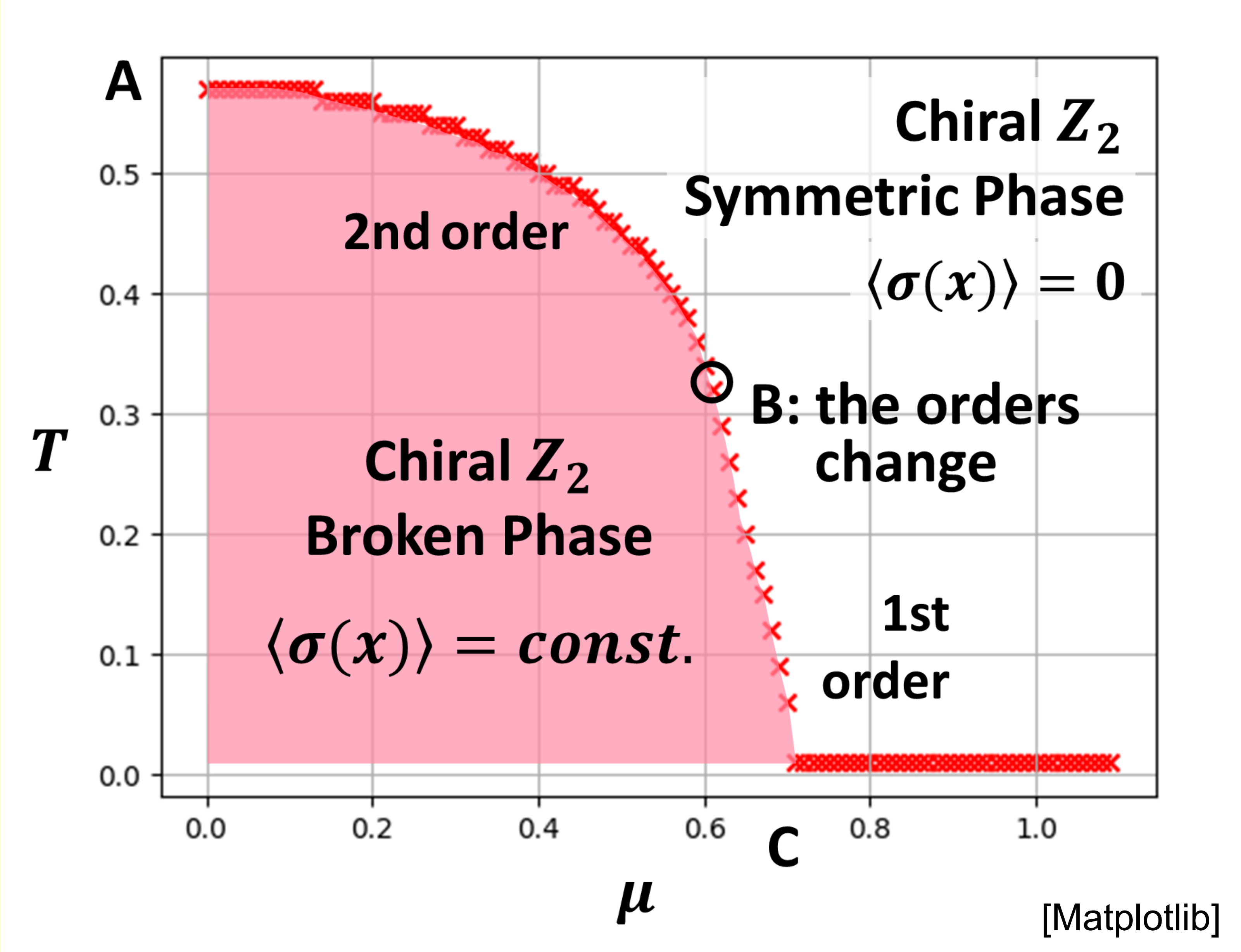
4. effective action

$$S_{eff} = \frac{1}{2g^2}\int \mathrm{d}^2x\,\sigma^2 - \log\det\left(\gamma_\mu\partial_\mu + \sigma + \mu\gamma_0\right)$$

$$\langle\sigma\rangle = \frac{1}{Z}\int \mathcal{D}\sigma \ e^{-N_f \ S_{eff}} \ \sigma$$

saddle point approx. $N_f \to \infty$ $\Longrightarrow$ **search for $\min_{\sigma,\beta,\mu} S_{eff}$**

Result 1 : $\langle\sigma(x)\rangle = constant$



Spatially varying order parameter Ref.[3,4]

- allow σ to depend on $\vec{x}$:

$$\sigma \to \sigma(\vec{x})$$

$$S_{eff}[\sigma,\beta,\mu] \to S_{eff}[\sigma(\vec{x}),\beta,\mu]$$

- self-consistency eq. (gap eq.)

$$\frac{\delta S_{eff}}{\delta\sigma(\vec{x})} = 0$$

assume $\quad \sigma(x) = \sigma(x+a); \quad a = L/N$

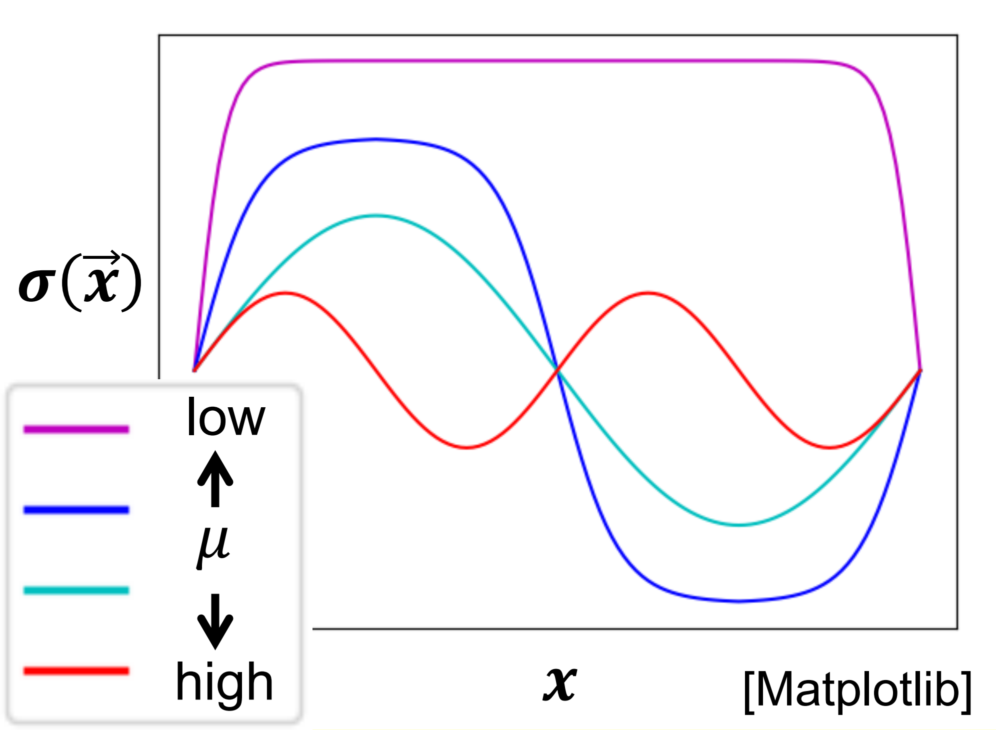
Fourier expansion

$$\sigma(x) = \sum_{m\in\mathbb{Z}} \tilde{\sigma}_m e^{i2q_m x} \; ; \quad q_m = \pi m/a$$

- solutions of the gap eq.

at high densities: only $\tilde{\sigma}_{\pm 1} \neq 0$

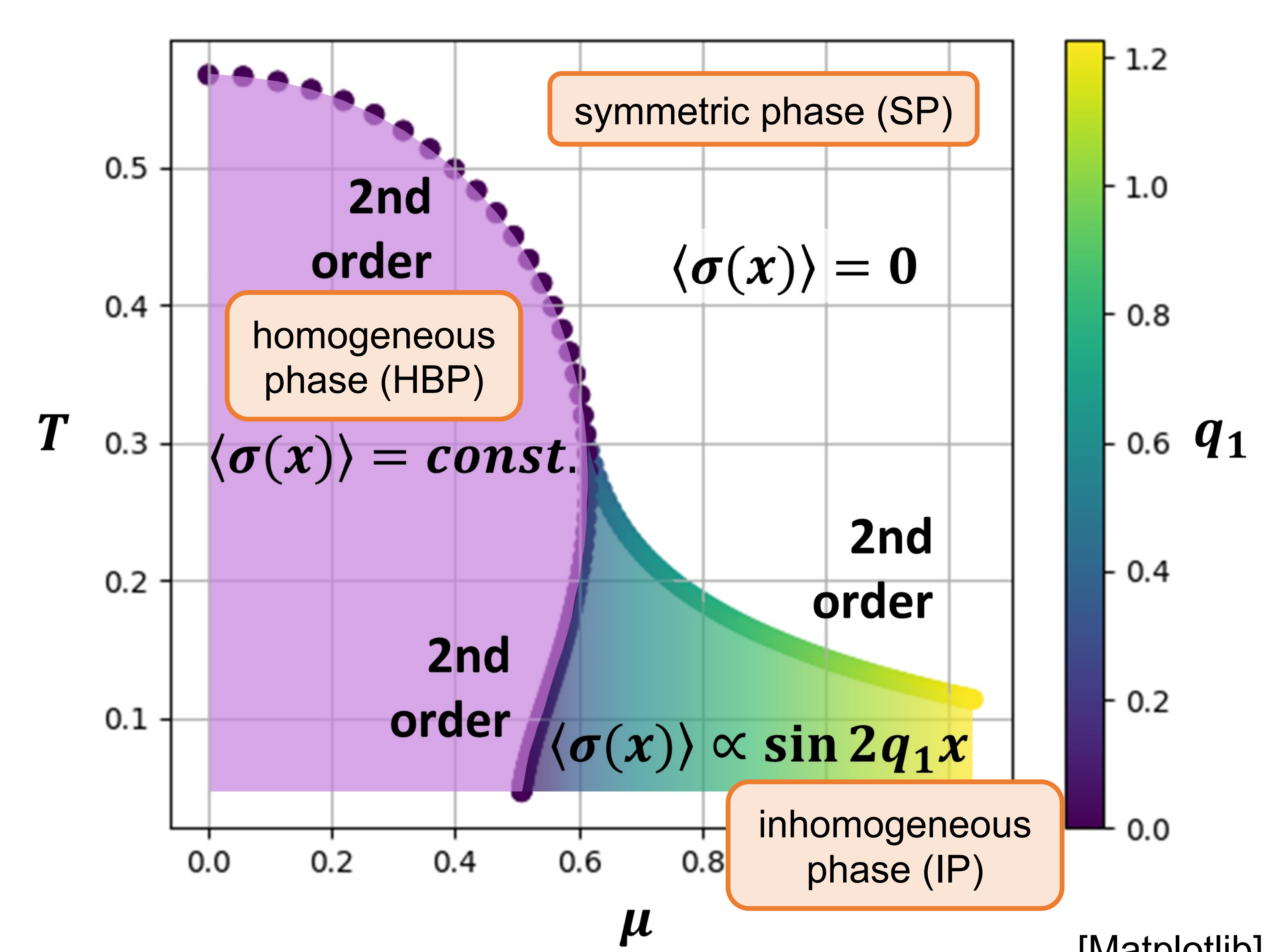
at low densities: $\tilde{\sigma}_{\pm 1}$ dominant



ignore higher-order Fourier coefficients

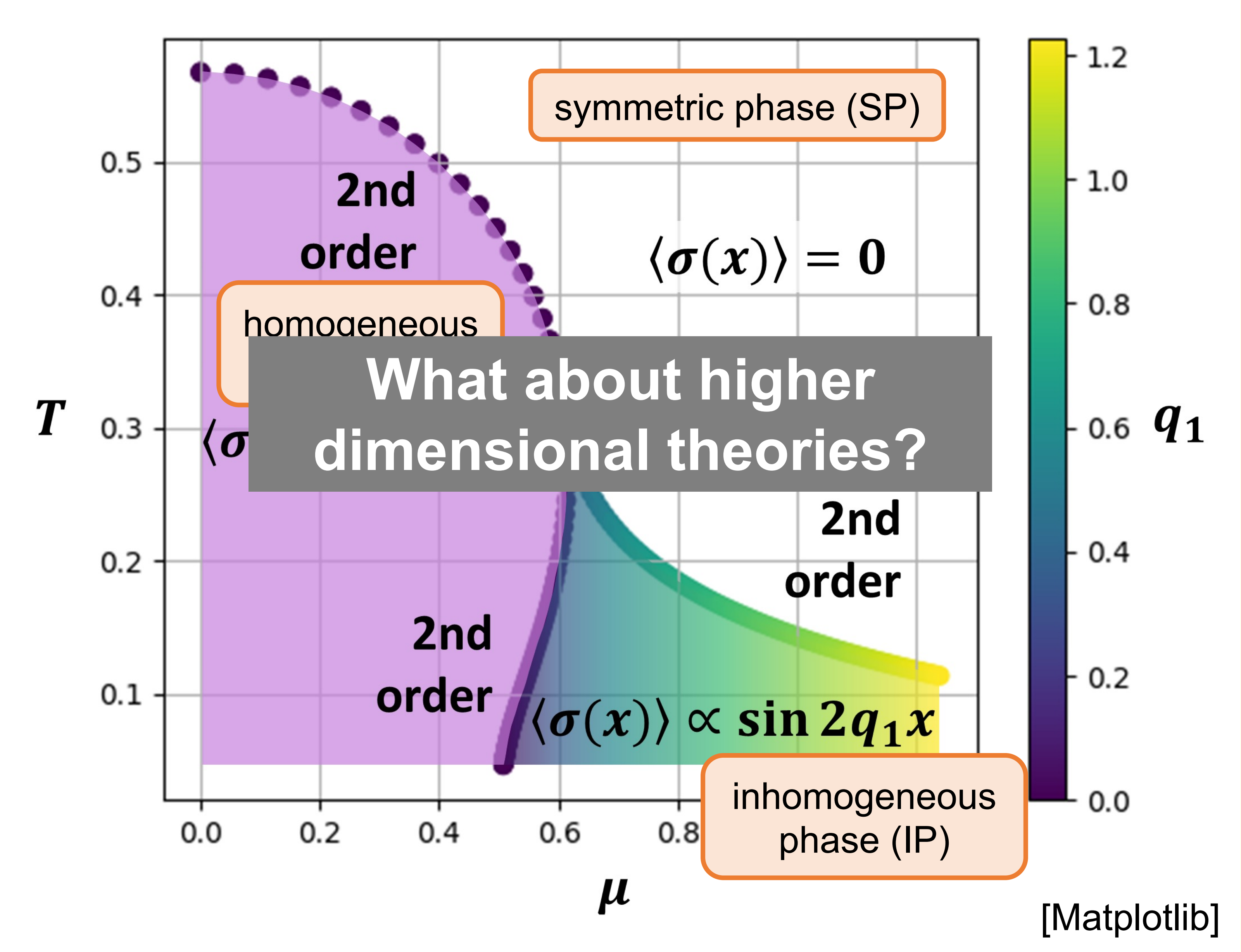
$$\Longrightarrow \quad \sigma(x) \propto \sin 2q_1 x$$

Result 2 : $\langle \sigma(x) \rangle \neq constant$



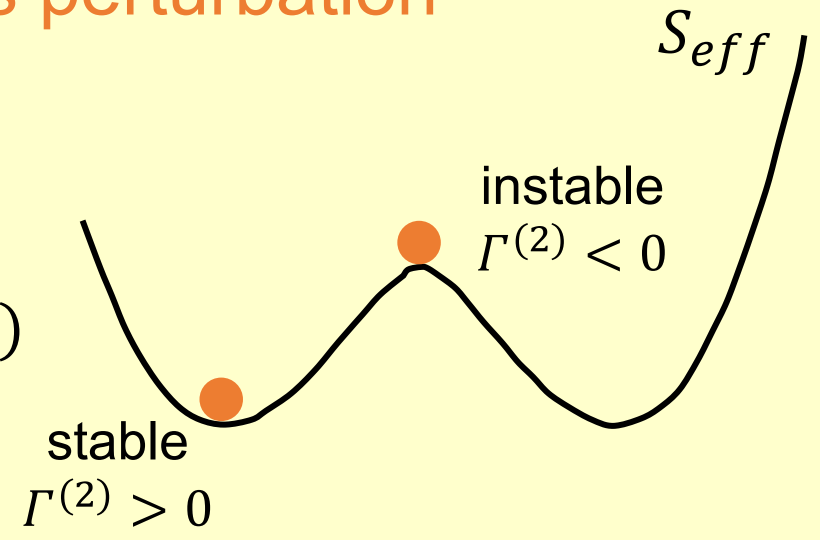
[Matplotlib]

Result 2 : $\langle \sigma(x) \rangle \neq constant$



Indirect detection of IP Ref.[5]

Stability analysis of HBP against inhomogeneous perturbation in $d+1$-dim. GN model



- effective action

$$S_{eff} = \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\, \sigma^2 - \log\det(\gamma_\mu \partial_\mu + \sigma + \mu\gamma_0)$$

- add an inhomogeneous perturbation

$$\sigma(const.) \to \sigma(\vec{x}) = \bar{\sigma}(const.) + \delta\sigma(\vec{x}),$$

$$D \to \bar{D} + \delta\sigma$$

$$D = \gamma_\mu \partial_\mu + \sigma + \mu\gamma_0$$

- (Taylor) expansion of S_{eff}

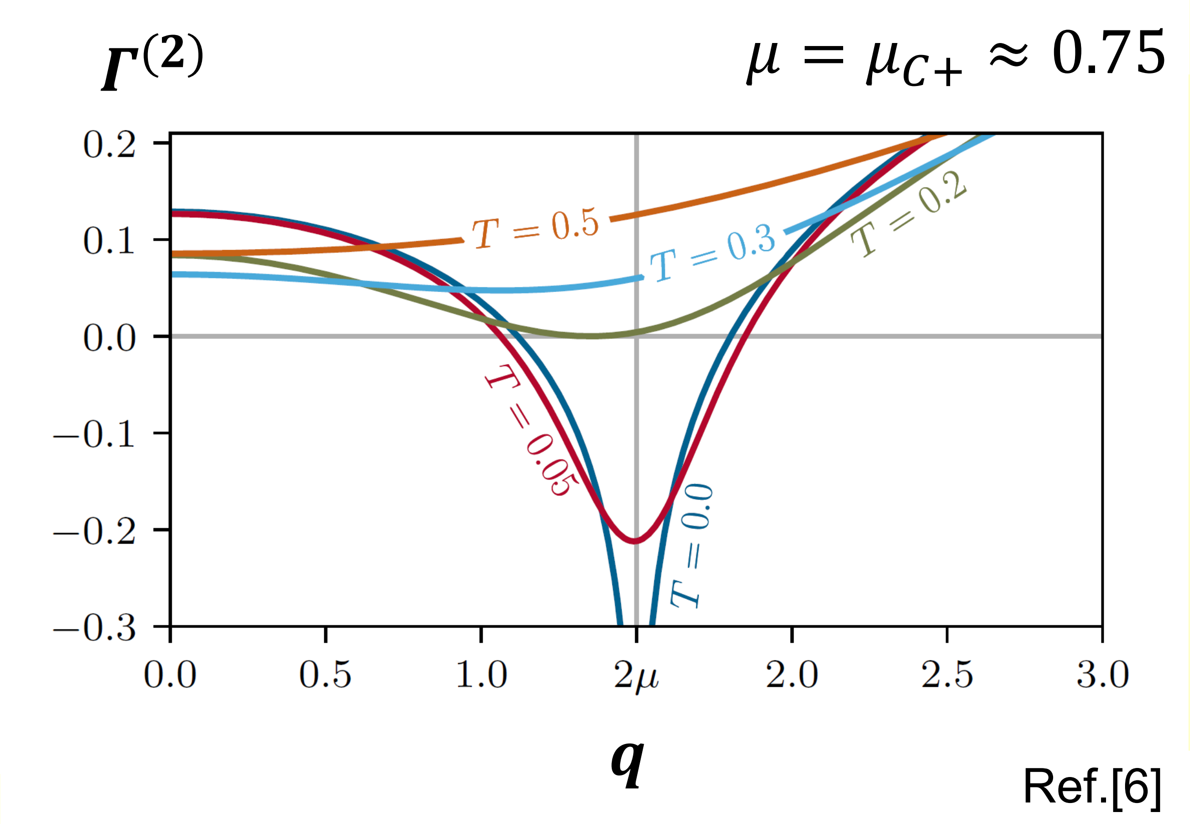
$$S_{eff} = S_{eff}^{(0)} + S_{eff}^{(1)} + S_{eff}^{(2)} + \cdots,$$

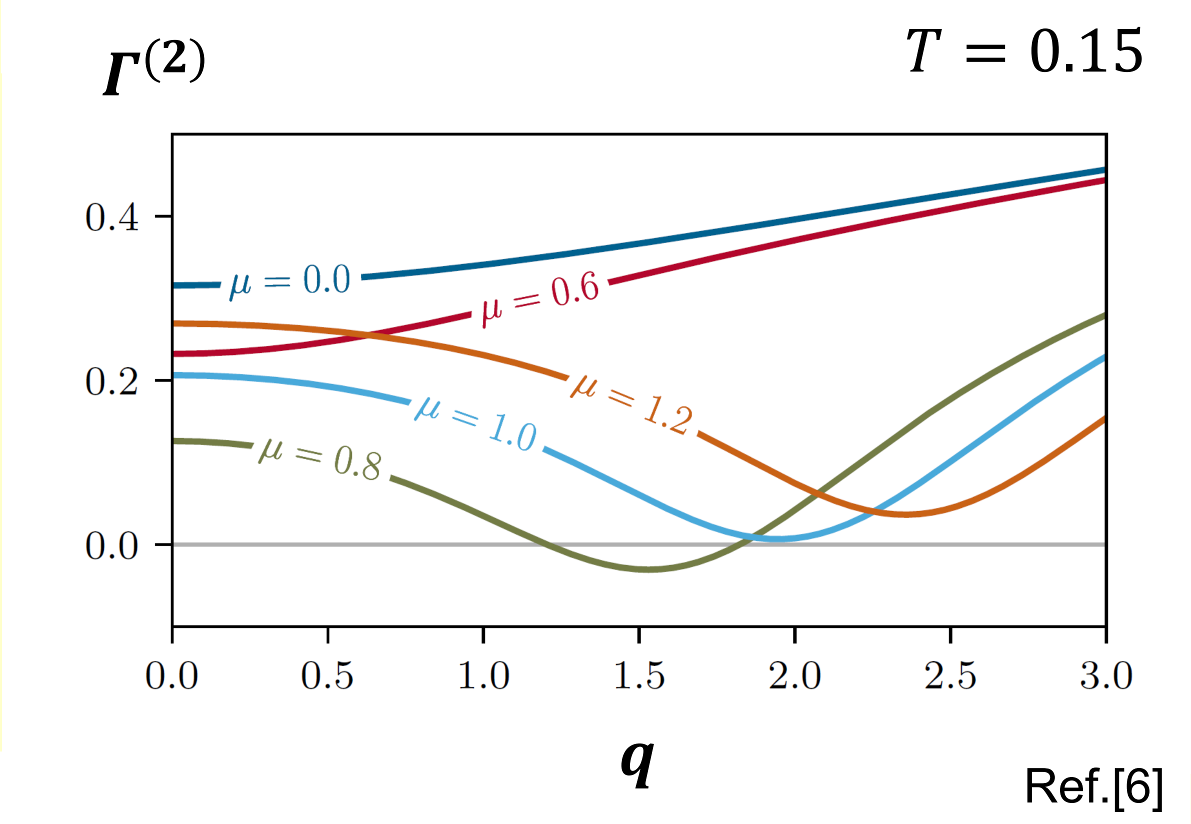
$$S_{eff}^{(2)} = \frac{\beta}{2}\int_{-\infty}^{\infty} \frac{dq}{2\pi}\, \Gamma^{(2)}\, \delta\tilde{\sigma}(q)^2$$

"curvature" of S_{eff}

Results of stability analysis ($d = 1$) Ref.[5]

curvature $\Gamma^{(2)}(\bar{\sigma}, \mu, T, q)$ $\begin{cases} < 0 \Leftrightarrow \text{instable HBP : imply IP} \\ > 0 \Leftrightarrow \text{stable HBP or SP} \end{cases}$



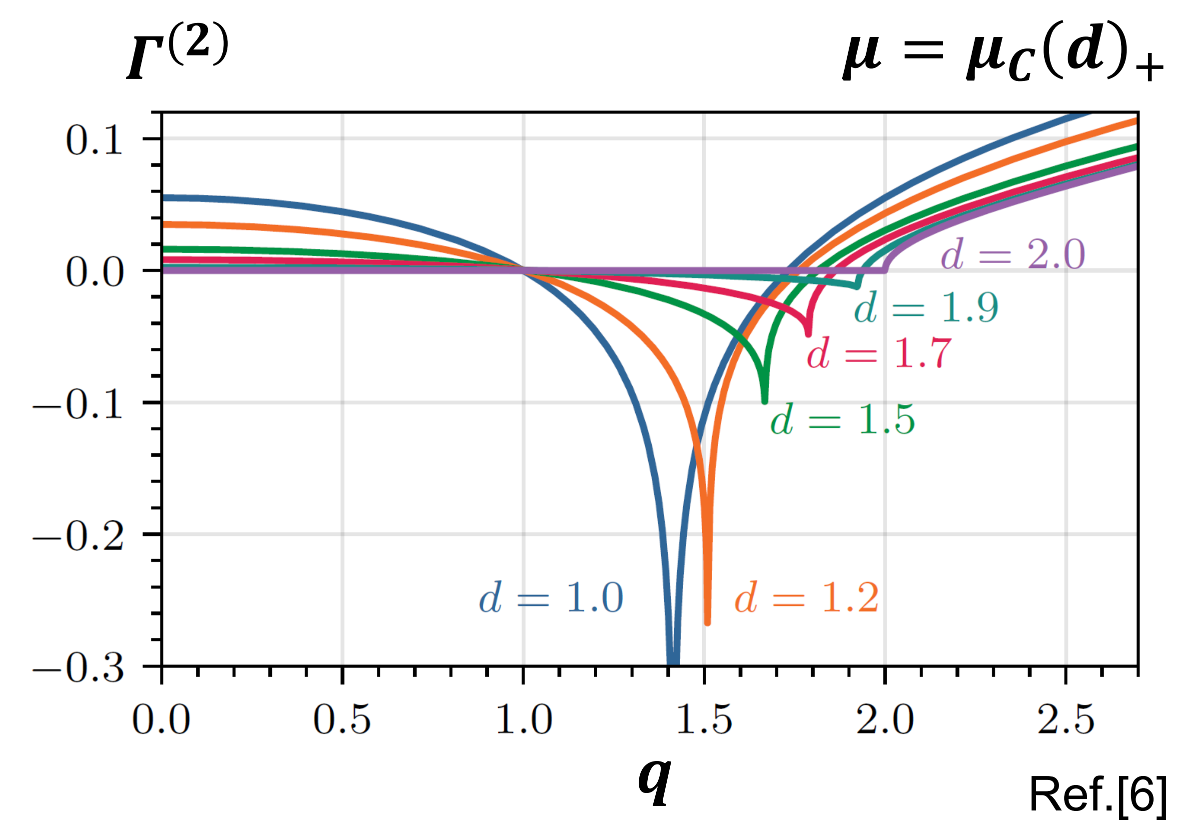


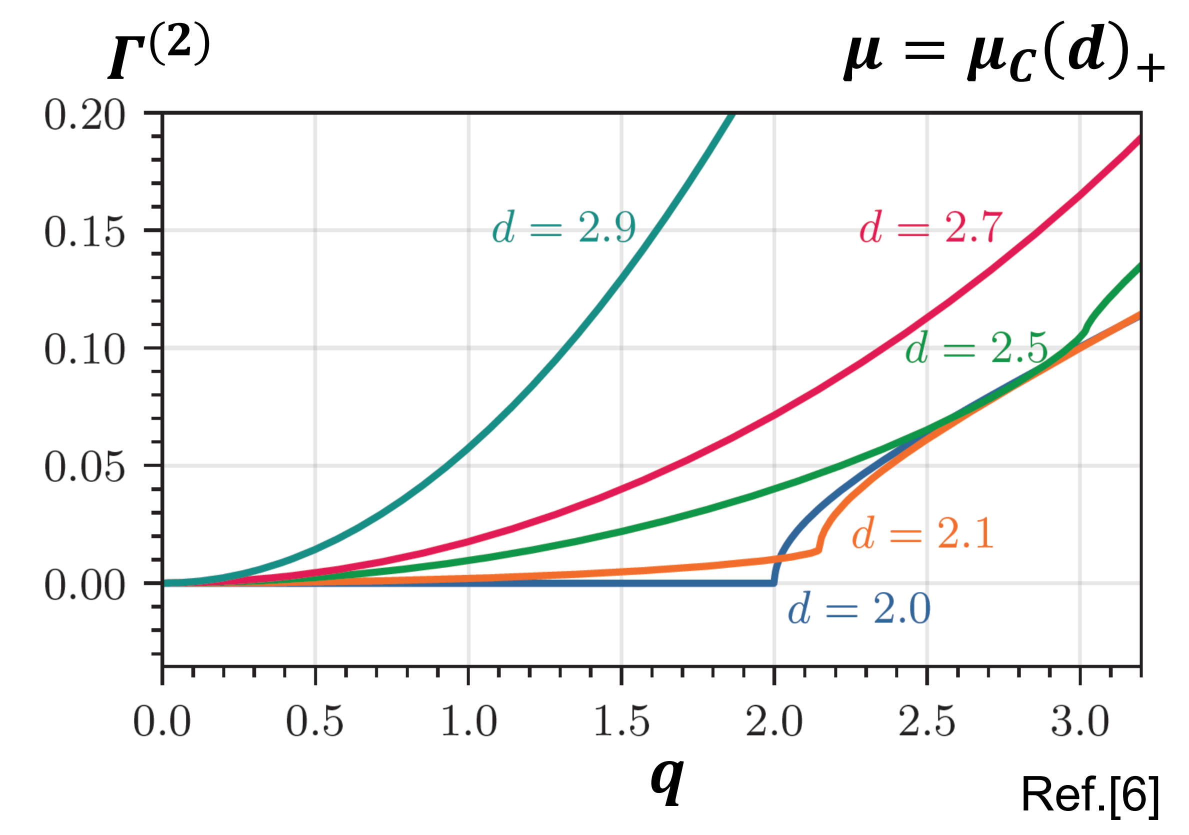
IP appears at very low T

IP appears when $\mu \sim 0.8$

IP in higher dimensions ($T = 0$) Ref.[6]

curvature $\Gamma^{(2)}(\bar{\sigma}, \mu, T, q)$ $\begin{cases} < 0 \Leftrightarrow \text{instable HBP : imply IP} \\ > 0 \Leftrightarrow \text{stable HBP or SP} \end{cases}$





$d < 2.0$: IP exists at higher densities

$d \geq 2.0$: IP does not exist

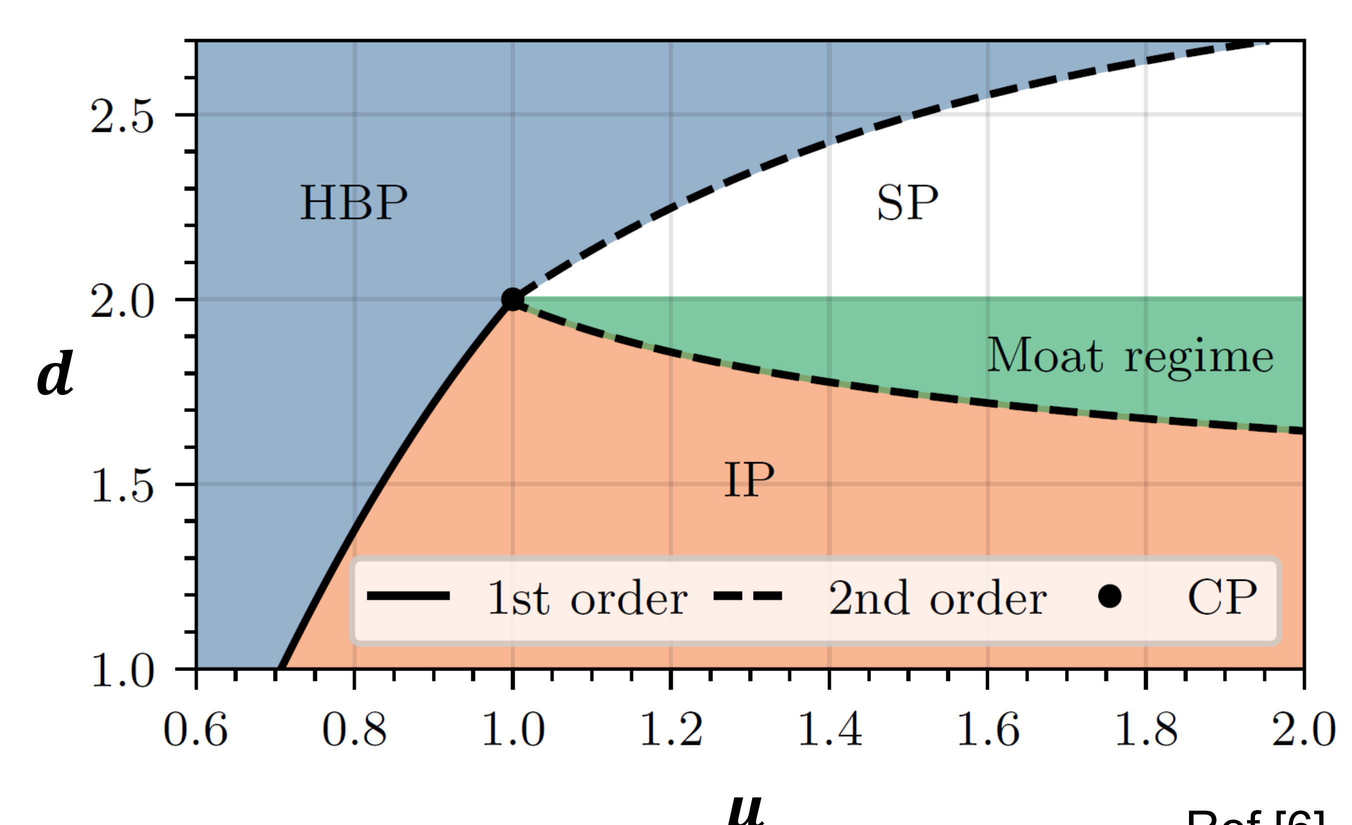
Summary and Prospect

$d+1$-dim. Gross-Neveu model
(toy model of QCD)

- SSB of chiral Z_2 symmetry
- HBP for all spatial dimensions d
- IP at high densities ($d \leq 2$)

"Moat regime"?

- "quantum pion liquid (QπL) phase" in QCD?



Ref.[6]

phase structure of the GN model at $T=0$

References

[1] W. Weise, *Nuclear Chiral Dynamics and Phases of QCD*, Progress in Particle and Nuclear Physics **67**, 299 (2012).

[2] D.J. Gross and A. Neveu, *Dynamical Symmetry Breaking in Asymptotically Free Field Theories*, Physical Review D **10**, 3235 (1974).

[3] J. Lenz *et al*. *Inhomogeneous Phases in the Gross-Neveu Model in 1+1 Dimensions at Finite Number of Flavors,* Physical Review D **101**, (2020).

[4] M. Thies and K. Urlichs, *Revised Phase Diagram of the Gross-Neveu Model*, Physical Review D **67**, (2003).

[5] A. Koenigstein *et al. Detecting Inhomogeneous Chiral Condensation from the Bosonic Two-Point Function in the (1 + 1)-Dimensional Gross–Neveu Model in the Mean-Field Approximation**, Journal of Physics A: Mathematical and Theoretical **55**, 375402 (2022).

[6] L. Pannullo, *Inhomogeneous Condensation in the Gross-Neveu Model in Noninteger Spatial Dimensions* $1 \leq D < 3$, Physical Review D **108**, (2023).

[7] B. Cowan, Topics in Statistical Mechanics (Second Edition) (World Scientific, 2021).

Thank you for your attention.

Questions in English or Japanese are welcome!

Introducing the auxiliary σ

$$Z \cdot 1 = Z \cdot \int \mathcal{D}\sigma \, \exp\left[-\int d^2x \, \frac{N_f}{2g^2}\sigma^2\right]$$

Gauss integration
with a normalization factor

$$= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \, \exp\left[-\int d^2x \left[\bar{\psi}i(\gamma_\mu\partial_\mu + \mu\gamma^0)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2 + \frac{N_f}{2g^2}\sigma^2\right]\right]$$

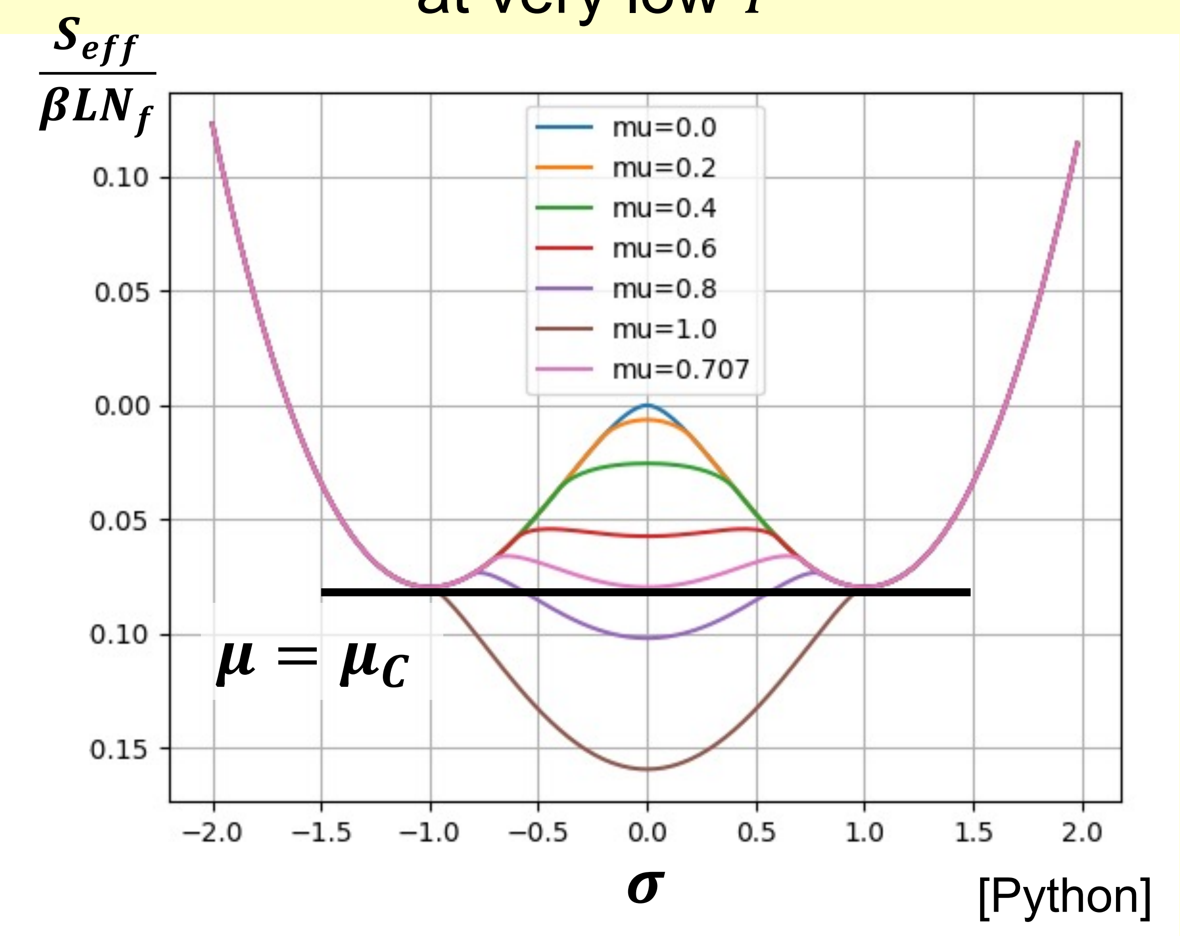
$$= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \, \exp\left[-\int d^2x \left[\bar{\psi}i(\gamma_\mu\partial_\mu + \mu\gamma^0 + \sigma)\psi + \frac{N_f}{2g^2}\sigma^2\right]\right]$$

$$\sigma \to \sigma' = \sigma - \frac{ig^2}{N_f{}^2}\bar{\psi}\psi$$

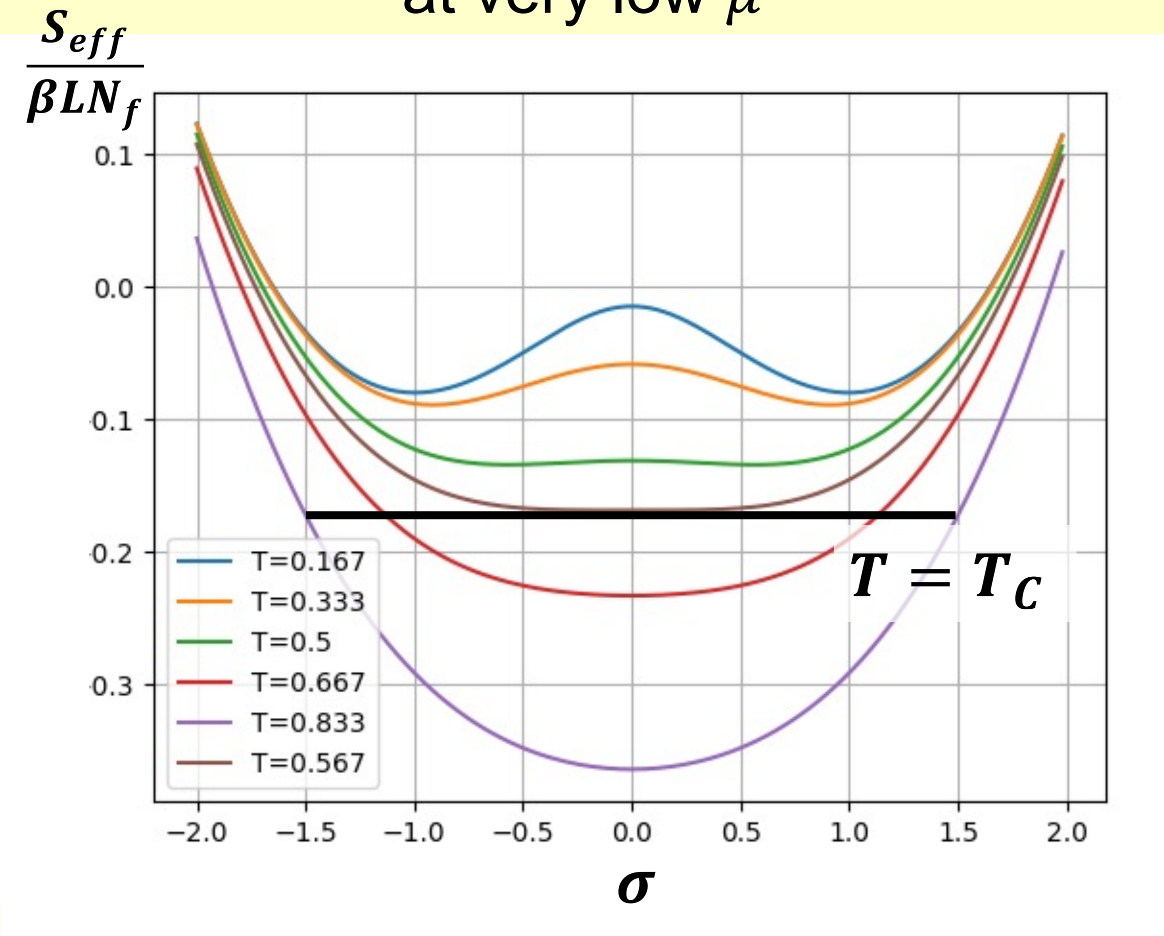
bilinear form w.r.t. $\bar{\psi}, \psi$
$\to \mathcal{D}\bar{\psi}\mathcal{D}\psi$ can be performed

Result 1 : effective potential $U_{eff} = S_{eff}/\beta L$

at very low T



at very low μ



minimum jumps: $\sigma = \pm 1 \to 0$

1st order phase transition

minimum gradually shifts: $\sigma = \pm 1 \to 0$

2nd order phase transition

Mermin-Wagner theorem Ref.[5]

For theories in d dimensions,

- $d \geq 3$

  Both continuous and discrete symmetries can be broken spontaneously.

- $d = 2$

  Only discrete symmetries can be broken spontaneously.
  ex.) the 2d Ising model

- $d = 1$

  Any symmetries cannot be broken spontaneously.

Result 1 : $\langle \sigma(x) \rangle = constant$

Mermin-Wagner theorem

Ref. [7]

in 2D theories:

✔ SSB of discrete sym.

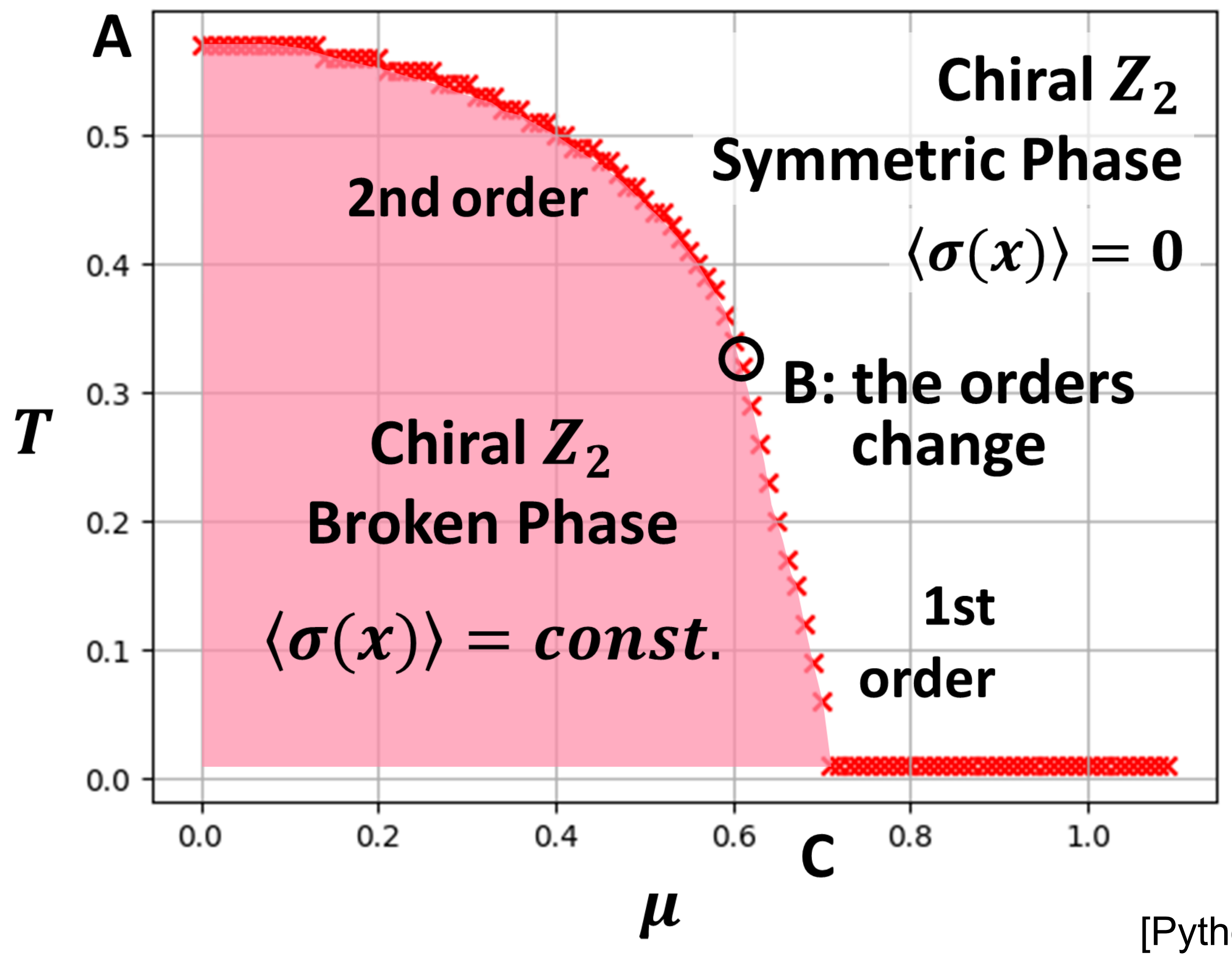
✘ SSB of continuous sym.

Z_2 symmetry: discrete

$$\psi \to i\gamma^5\psi, \qquad \bar{\psi} \to i\bar{\psi}\gamma^5$$

$$\Downarrow$$

phase transition is allowed



[Python]

Result 2 : $\langle \sigma(x) \rangle \neq constant$

Mermin-Wagner theorem

Ref. [7]

in 2D theories:

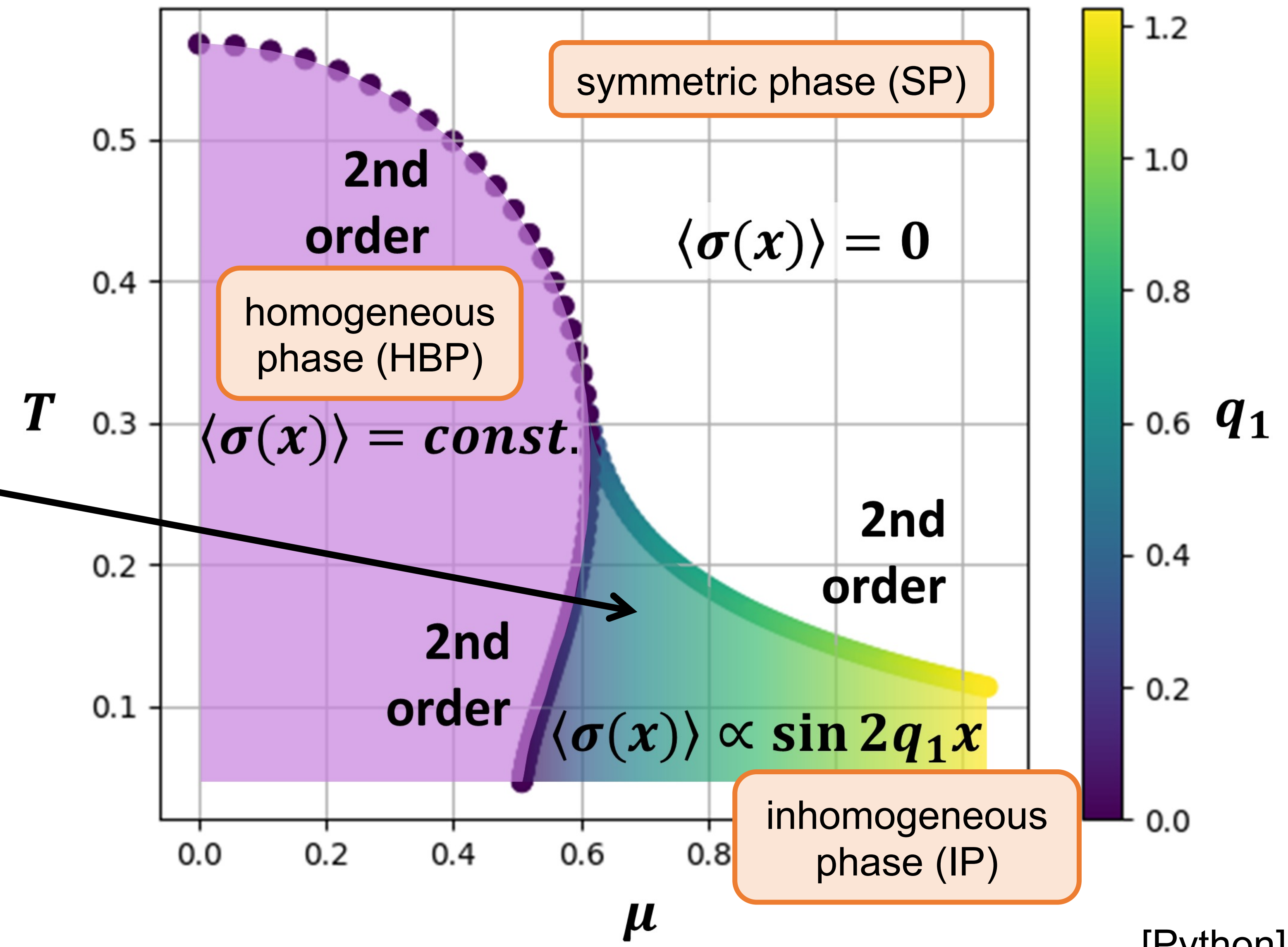
- ✔ SSB of discrete sym.
- ✘ SSB of continuous sym.

SSB of translational sym.?
(continuous)

Large N_f → SSB is allowed

so far: my graduate thesis

What about higher dimensional theories?



[Python]

$d+1$-dim. Gross-Neveu model Ref.[6]

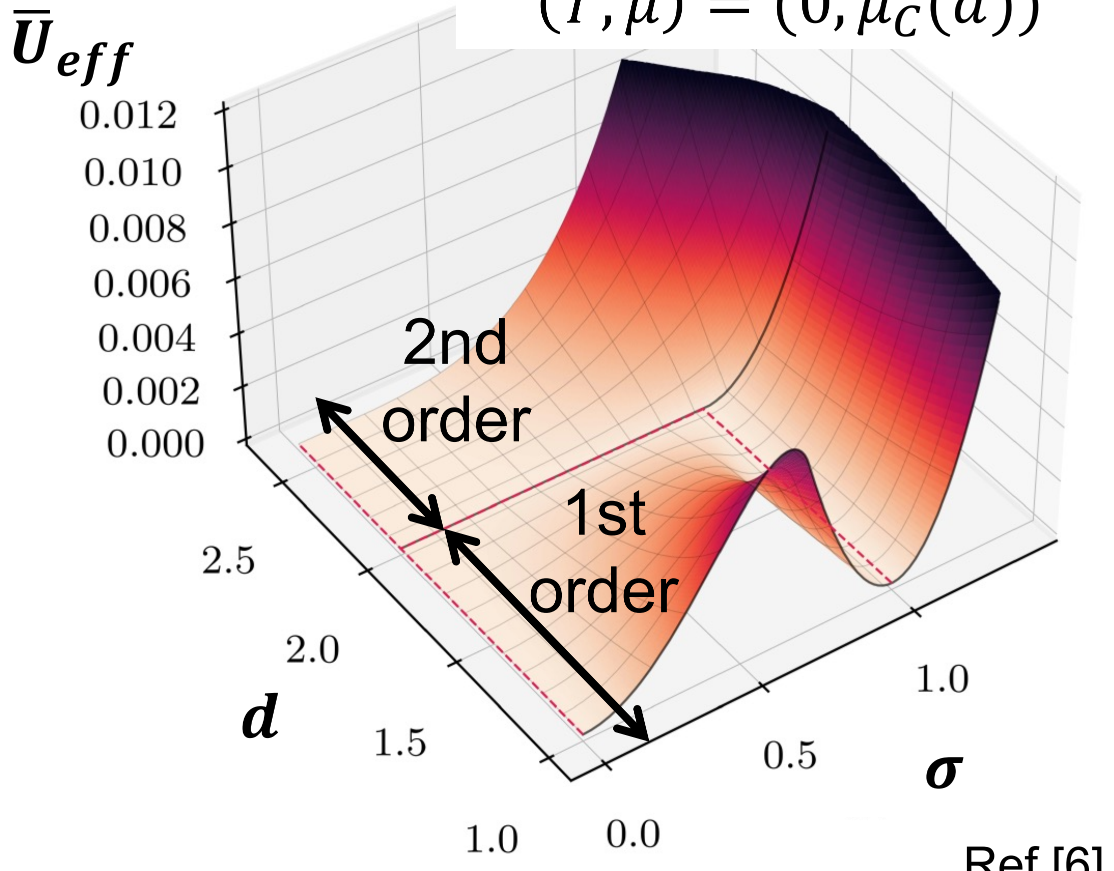
$d+1$-dim. Gross-Neveu effective action (Euclidean spacetime)

$$S_{eff} = \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\, \sigma^2 - \log\det(\gamma_\mu \partial_\mu + \sigma + \mu\gamma_0)$$

dimensional renormalization + assume σ: x-independent

renormalized effective action ($T=0$)

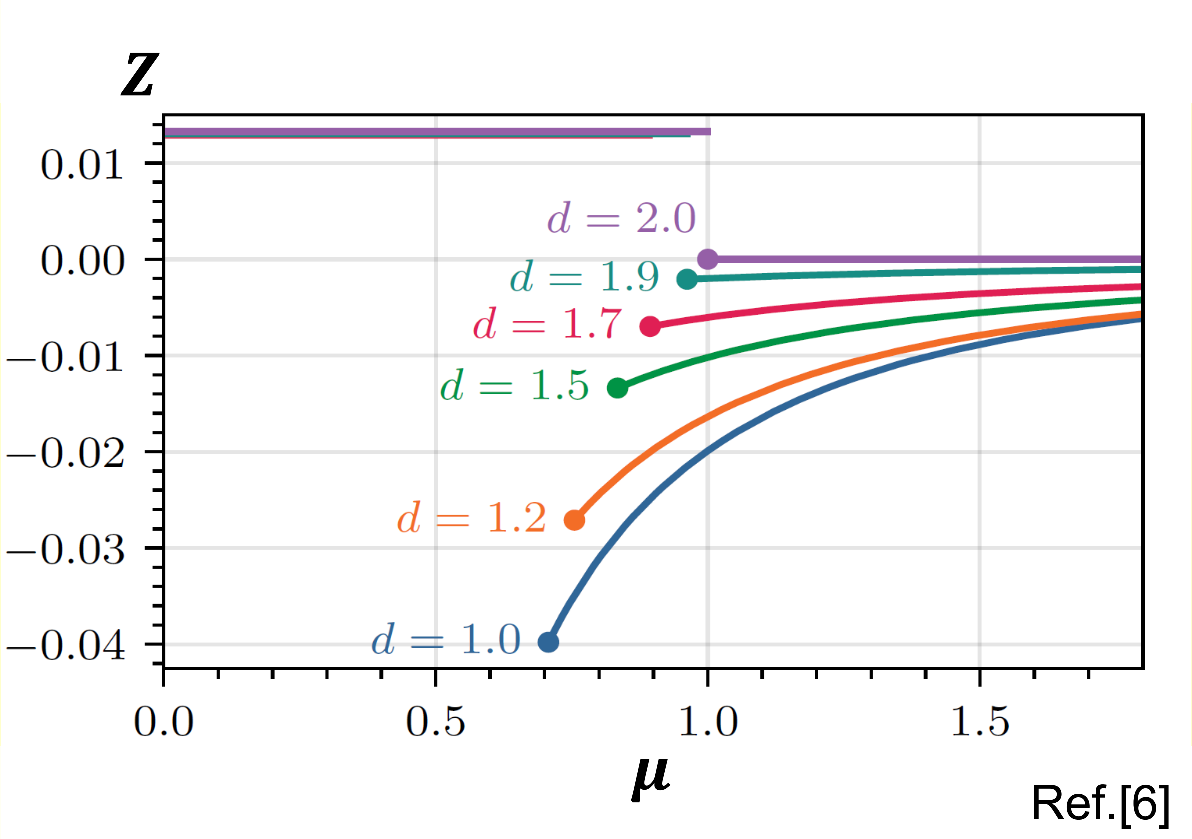
$$\bar{S}_{eff} \propto \bar{U}_{eff}[\sigma, \mu, d]$$



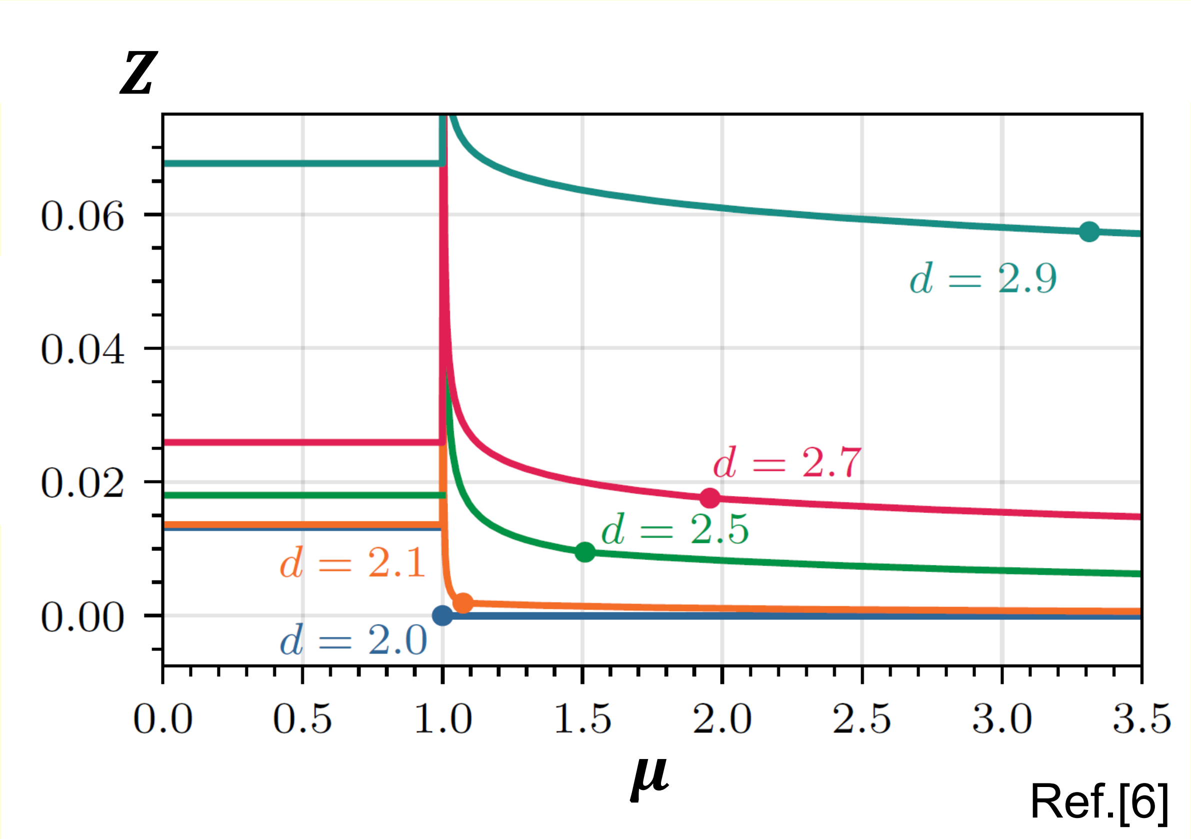
Ref.[6]

Existence of the moat regime ($T = 0$) Ref.[6]

$$Z = \frac{1}{2}\frac{d^2\Gamma^{(2)}}{dq^2} \begin{cases} < 0 \Leftrightarrow \text{moat regime} \\ > 0 \Leftrightarrow \text{not moat regime} \end{cases}$$



$d \leq 2.0$



$d \geq 2.0$